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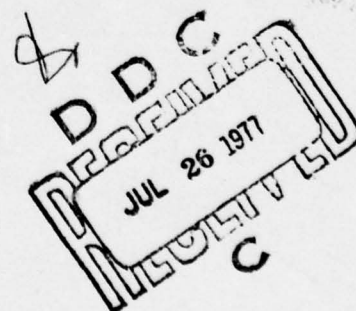
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A CUMULATIVE DAMAGE THEORY

OF FATIGUE FAILURE

by

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A CUMULATIVE DAMAGE THEORY OF FATIGUE FAILURE

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ABSTRACT

A rational phenomenological theory of fatigue lifetime prediction under arbitrary variation of cycle amplitude is presented. The theory is based on the concept of damage curve families and on an equivalent loading postulate which defines specimens that have suffered identical damage under different loading programs. Lifetime analysis has been performed for various cases of piecewise constant and continuous variation of cycle amplitude. For continuous variation, the method requires numerical integration of nonlinear first order differential equations which have been established in this work.

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1. Introduction

A basic problem in design for fatigue safety is the prediction of lifetime of a structural part when the amplitude of cyclic load varies in prescribed fashion with the number of cycles.

The classical test for fatigue failure is sinusoidal constant amplitude cycling-to-failure of a specimen or structural part resulting in the number of cycles-to-failure or lifetime $N(\sigma)$, where σ is the stress amplitude. The plot of σ against N is known as the S-N curve. Suppose now that the specimen is subjected to a loading program $\sigma(\bar{n})$ where $\bar{n}$ is number of cycles. The problem is to determine $\bar{N}$ - the number of cycles to failure under this loading program.

In view of the extreme complexity of internal fatigue failure mechanisms, there is little hope of resolving the problem on the basis of micro-structural considerations. An alternative is to consider the problem in phenomenological fashion. It should, however, be borne in mind that fatigue failure test data are subject to considerable scatter. In order not to further complicate a very difficult problem, it is reasonable to disregard initially the statistical scatter aspects of the problem. Thus all results are to be interpreted in some "central measure" sense. The problem considered in this sense has become known as cumulative damage theory. It has been the subject of numerous investigations, particularly in the last thirty years. Since there are many excellent reviews of previous work, e.g. [1, 2, 3, 4], present discussion will be limited to some selected aspects.

A basic concept in the approaches given is the damage function which defines in some sense the damage produced in a specimen when subjected to n cycles at stress amplitude σ. This function is often written as $F(n/N)$ where N is the lifetime for constant amplitude σ cycling. Past work seems to have been limited

to the case where the loading is piecewise constant, here referred to as multi-stage loading. Particular attention has been given to the simplest case of two-stage loading. For multi-stage loadings the damage function is written

$$F(\frac{n_1}{N_1}, \frac{n_2}{N_2}, \ldots \frac{n_k}{N_k}) \qquad (1.1)$$

It is stipulated that

$$0 \le F \le 1 \qquad (1.2)$$

failure being defined by F=1.

A simplistic and well known damage function has been postulated by Palmgren [5] and by Miner [6] and has become known as Miner's rule. According to them, failure in a multi-stage loading is defined by

$$\sum_i \frac{n_i}{N_i} = 1 \qquad (1.3)$$

where

n_i - number of cycles at level σ_i

N_i - lifetime at constant amplitude σ_i

According to this rule damage produced by σ_i for n_i cycles is defined as n_i/N_i and the individual damages are additive and independent of sequence.

This simplistic approach does not in general comply with reality. It is known experimentally and it is physically plausible that the order of loading can significantly affect the lifetime of a specimen. Thus, in two-stage constant amplitude loadings, the left side of (1.3), which is sometimes termed Miner's coefficient,

is usually different from unity and is dependent on loading sequence.

It should be noted that the only material characteristic entering into Miner's rule is the S-N curve for constant amplitude loading. It seems unreasonable to assume that such simple information would be sufficient for lifetime prediction under arbitrary loading programs.

Marco and Starkey [7] assumed that "damage" produced by n cycles at level σ is given by $(n/N)^\alpha$ where the exponent is a function of σ. They arrived at a failure condition which for two-stage loading assumes the form

$$\left(\frac{n_1}{N_1}\right)^{\alpha_1/\alpha_2} + \frac{n_2}{N_2} = 1 \tag{1.4}$$

and can be generalized to multi-stage loading. Apparently, the dependence of the exponent α on σ must be uncovered by experimental means and this does not appear to be an easy task. It will be seen that (1.4) is a specific special case of the theory to be developed in this work which does specify the dependence of α on σ in unequivocal fashion.

Shanley [8] has constructed a theory in which it is postulated that damage can be described in terms of $C\sigma^\alpha n$. Corten and Dolan [9] described damage in the form Cn^α. A detailed account of these and other approaches may be found in [1].

The present work provides a systematic new approach to the problem which is based on the concept of damage curve families, to be defined further on. The description of "damage" by some function is avoided. A better concept is residual lifetime, this being a quantity which, unlike damage, can be defined and measured. The damage curves are defined in terms of this quantity.

2. The Concept of Damage Curves

We consider a specimen which is subjected to cyclic loading of constant maximum amplitude σ_1, failure taking place after N_1 cycles. The pair of numbers N_1 ; σ_1 define a point on the S-N curve of the specimen, Fig. 1.

If the specimen is subjected to n_1 cycles, $n_1 < N_1$, at σ_1, it suffers a certain amount of unknown internal damage. The pair of numbers, n_1 ; σ_1 , defines a point in the s-n plane. The region bounded by the n, σ axes and the S-N curve may be considered as a damage region. Each of the points within the damage region specifies in some sense an amount of damage for n cycles at amplitude σ .

Next we consider the situation where after n_1 cycles at σ_1, cycling is continued at some stress amplitude σ. The remaining number of cycles to failure is called the residual lifetime and is denoted $N-n$ where N is the constant amplitude lifetime at σ. The pairs of numbers $n; \sigma$ define a curve in the S-N plane, Fig. 1, which is called the damage curve. Evidently, the damage curve under consideration must pass through the point n_1 ; σ_1.

It should be noted that in this description the damage suffered by a specimen is essentially defined by its residual lifetime which is a measurable quantity. The S-N curve is the ultimate damage curve when remaining lifetime is zero.

The basic assumption is made that a damage curve is uniquely defined by its initial point. This implies that if any other point on the damage curve $n_2; \sigma_2$, say, is interpreted as the first stage of a two-stage cycling experiment, the remaining lifetimes for various subsequent stages will be determined by the initial damage curve which passes through $n_1; \sigma_1$. This assumption can be accepted with or without distinction between low-high and high-low loadings. Further discussion of this latter aspect is given later.

Uniqueness of the damage curve passing through a point and subsequent analytical procedures for lifetime computations are based on a fundamental postulate which will now be explained. For this purpose we shall introduce the concept of equivalent cyclic loadings. Cyclic loading programs are termed equivalent for stress amplitude σ if for subsequent constant amplitude loading, at same stress level σ, residual lifetimes are the same. The equivalent loading postulate is now stated as follows: Cyclic loadings which are equivalent for one stress level are equivalent for all stress levels.

To show uniqueness of damage curves on the basis of this assumption, consider the damage curve, Fig. 1, as having been obtained by a series of two-stage loadings in all of which the first stage is n_1, σ_1, as previously explained. In the two stage program σ_1 for n_1 cycles, σ_2 for an infinitesimal number of cycles the residual lifetime is n_2^r. In the loading program σ_2 for $N_2 - n_2^r$ cycles the residual lifetime is evidently also n_2^r. Therefore these two loadings are equivalent. It follows by the equivalent loading postulate that if after completion of these two loading programs the specimens are cycled to failure at some other stress level, σ, the residual lifetime will be the same n^r, Fig. 1. Therefore the loadings σ_1 for n_1 cycles, σ for infinitesimal number of cycles; σ_2 for $N_2 - n_2^r$ cycles, σ for infinitesimal number of cycles have the same residual lifetime n^r for subsequent constant σ cycling to failure. This proves uniqueness of the damage curve.

A damage curve as defined describes the results of a large number of fatigue failure experiments in two level loadings. It is desirable that a maximum number of parameters which influence fatigue failure be kept constant in this series of experiments. The parameters to be considered are : frequency of loading, mean stress σ_m - the average of maximum and minimum amplitudes in a cycle and R - the ration between minimum and maximum amplitudes in a cycle. It may be assumed that the frequency is constant in all cyclic loadings and that either σ_m or R is

constant. In the special case of $\sigma_m = 0$, $R = -1$ both σ_m and R can be kept constant. It is in general to be expected that a change in any of these parameters will change the damage curves.

Because of the inherent scatter of fatigue testing, all damage curves must be interpreted in some best fit sense, as is the case for S-N curves.

The equations of the damage curves are written in the general form

$$\sigma = \sigma(n,\gamma) = \sigma_s f(n,\gamma) \qquad (2.1)$$

where γ is a parameter, σ_s is the static failure stress or some other convenient stress parameter and f is a non-dimensional function. We now proceed to discuss some general properties of the damage curves.

(a) All damage curves pass through the point 0; σ_s.

Evidently, since static failure occurs at one quarter cycle which is considered as $n=0$, it follows from (2.1) that

$$f(0,\gamma) = 1 \qquad (2.2)$$

However, curve fitting difficulties may require use of a σ_s value which is not equal to the static test result.

(b) Damage curves do not intersect.

To prove this assertion we consider the case of two low-high load cycling programs with equal amplitude pairs σ_1 and σ_2. The number of cycles at σ_1 is n_1 in the first program and $n_1' > n_1$ in the second program. Obviously the remaining lifetime $N_2 - n_2$ at σ_2, in the first program must be larger than the remaining lifetime $N_2 - n_2'$ in the second program. Consequently,

$$n_2' > n_2 \tag{2.3}$$

Fig. 2a shows a hypothetical intersection between two damage curves. $A_1 A_2$ and $B_1 B_2$ represent the two loading programs described above. It is seen that (2.3) is violated. Therefore, intersection is not possible and neighboring damage curves are as in Fig. 2b.

(c) Damage curves do not intersect the n or σ axis (except at $\sigma = \sigma_s$).

Suppose that a damage curve intersects the n axis. This would imply that in a two-stage loading program with amplitudes $0, \sigma$ the lifetime is smaller than $N(\sigma)$. But the lifetime must be $N(\sigma)$ since the loading is really one stage. Therefore the damage curve must pass through the point $0, \sigma$ which is not possible in view of (a).

Suppose that a damage curve passing through n_1, σ_1 intersects the σ axis at σ_2. This would imply that if the specimen is exposed to n_1 cycles at σ_1 the remaining lifetime for cycling at σ_2 is N_2, which can only be correct if $n_1 = 0$. Therefore, no damage curve intersects the σ axis below σ_s.

In view of the properties (a) - (c) of the damage curves, it is seen that they form a family of non-intersecting curves within the region of the s-n plane which is bounded by the n, σ axes and by the S-N curve. The damage curves all pass through the point $0; \sigma_s$ and approach the n, σ axes asymptotically. The S-N curve is the limiting damage curve. It corresponds to a two stage loading in which failure occurs for the first stress σ_1 and residual lifetime for σ_2 is zero.

The simplest procedure to assess the form of the damage curves is to assume that their mathematical description is similar to that of the S-N curve. Let the equation of the S-N curve be represented in the form

$$\sigma = \sigma_s f(N, \Gamma) \qquad (2.4)$$

where N denotes value of n for a point on the S-N curve. Then the equation of the damage curves is assumed to be given by the similar form

$$\sigma = \sigma_s f(n, \gamma) \qquad (2.5)$$

The parameter γ for a damage curve passing through the point $n_1; \sigma_1$ is determined by inserting those values into (2.5) and solving for γ.

The two most common analytical representations of S-N curves are:

(1) σ is a linear function of $\log_{10} N$. This will be called the <u>semi-log</u> form.

(2) $\log_{10} \sigma$ is a linear function of $\log_{10} N$. This will be called the <u>log-log</u> form. Thus

$$\sigma = \sigma_s (1 + \Gamma \log N) \qquad (2.6)$$

or

$$\sigma = \sigma_s N^{\Gamma}$$

In both of these representations $N=1$ is considered physically equivalent to $N=0$.

It follows that the corresponding forms of damage curves passing through a specified point $n_1; \sigma_1$ are:

$$\sigma = \sigma_s (1 + \gamma \log n) \qquad \text{(a)}$$

$$\qquad\qquad\qquad\qquad\qquad (2.8)$$

$$\gamma = \frac{\sigma_1/\sigma_s - 1}{\log n_1} \qquad \text{(b)}$$

$$\sigma = \sigma_s n^\gamma \; ; \quad \log(\sigma/\sigma_s) = \gamma \log n \qquad \text{(a)}$$

$$\gamma = \frac{\log(\sigma_1/\sigma_s)}{\log n_1} \qquad \text{(b)} \qquad\qquad (2.9)$$

The form (2.8) is generally more suitable for fiber composites while the form (2.9) is more suitable for metals.

The analytical expressions (2.6-2.9) are merely convenient approximations and should not be taken too literally. It is seen that all curves intersect the N axis contrary to theorem (c). The conclusion is that for very low stress amplitudes the analytical expressions are not good approximations.

Other fitting difficulties may occur at small number of cycles. It frequently happens that the σ_s determined from S-N curve fitting is not the σ_s found by static testing. In that case the former should be chosen and be regarded as a curve fitting parameter.

It is permissible to fit a convenient portion of the S-N curve, for the range $\sigma_1 \leq \sigma \leq \sigma_2$, say, by the analytical expressions. In that case, the validity of the damage curve expressions is limited to that loading range and σ_s is a fitting parameter.

The S-N curve may be represented by more than one function. For example: a broken straight line in log-log presentation. In that case the damage curves would be similarly represented.

If the analytical forms do not prove accurate enough it is necessary to determine the damage curves by testing. In order to determine a damage curve which passes through the point $n_1; \sigma_1$ a group of specimens are subjected to n_1 cycles at σ_1 and are then cycled to failure at different stress levels σ_2, σ_3, ... Each of the lifetimes at σ_2, σ_3, ... determine a point on the damage curve.

The damage curves can be separately defined for two stage loadings which are of the low-high type or of the high-low type. There then arises the question if both sets of curves are identical. Consider for this purpose a loading program in which n_1 cycles take place at σ_1. Then the amplitude is raised to σ_2 for the duration of a very small number of cycles. After that, the amplitude is reduced again to σ_1, and cycling is continued until failure. A diagram of such a loading program in the s-n plane is shown in Fig. 3. If the number of cycles to failure is insignificantly affected by the rise to σ_2, and is thus equal to N_1, the lifetime at σ_1, then the segment A_1A_2 is insignificant and the upward damage curve A_1B_1 coincides with the downward damage A_2B_1. If, however, this is not the case, the upward and downward damage curves are different and the damage curves exhibit a hysteresis phenomenon. On the basis of experience in fatigue testing, it is reasonable to assume that such an effect will not be significant at low to moderate stress rises but may be significant at high stress rises with consequent conclusions for the identity of upward and downward damage curves.

It is to be emphasized that it does not seem possible to include such hysteresis effects in the simple analytic representations (2.8 - 2.9) since they are based on the form of the S-N curve where hysteresis cannot enter. Therefore, such effects must be uncovered by experiments. This appears to be a difficult task because of the considerable scatter of fatigue failure results.

3. Analysis of Loading Programs

The simplest loading program is two-stage cyclic loading. Its analysis in terms of the damage curves is trivial since the damage curves have been defined in terms of residual lifetime under such loadings. Let σ_1 be applied for n_1 cycles. Then the stress amplitude is changed to σ_2 and cycling is continued until failure occurs after n_2 cycles. It is required to compute n_2.

Let it be assumed that the S-N curve is of form (2.6) and that the damage curves are of semi-log form (2.8a). Application of n_1 cycles at σ_1 defines a damage point, Fig. 1. The damage curve passing through this point is given by (2.8). It follows that

$$\sigma_2 = \sigma_s [1 + \frac{\log(N_2 - n_2)}{\log n_1} (\frac{\sigma_1}{\sigma_s} - 1)] \tag{3.1}$$

Solving for n_2 we have

$$n_2 = N_2 - n_1^{\frac{1-s_2}{1-s_1}} \tag{3.2}$$

where

$$s_1 = \frac{\sigma_1}{\sigma_s} \qquad s_2 = \frac{\sigma_2}{\sigma_s} \tag{3.3}$$

The quantity

$$M = \frac{n_1}{N_1} + \frac{n_2}{N_2} \tag{3.4}$$

is defined as Miner's coefficient (for two stage loading). It follows easily from (2.6a) and (2.8a) that (3.2) can be written as

$$(\frac{n_1}{N_1})^{\frac{1-s_2}{1-s_1}} + \frac{n_2}{N_2} = 1 \tag{3.5}$$

which replaces Miner's rule for the present case. It follows from (3.4 - 5) that

$$M = 1 + \frac{n_1}{N_1} - (\frac{n_1}{N_1})^{\frac{1-s_2}{1-s_1}} \tag{3.6}$$

It is easily shown that

$$\begin{array}{lll} M \geq 1 & & s_1 \geq s_2 \\ & \text{when} & \\ M \leq 1 & & s_1 \leq s_2 \end{array} \tag{3.7}$$

A similar analysis for damage curves of log-log type, (2.9), yields the results

$$(\frac{n_1}{N_1})^{\log s_2/\log s_1} + \frac{n_2}{N_2} = 1$$

$$M = 1 + \frac{n_1}{N_1} - (\frac{n_1}{N_1})^{\log s_2/\log s_1} \tag{3.8}$$

Again, it is easily shown that (3.8) complies with (3.7).

Note that (3.5) and (3.8) are of the form (1.4) which is based on an exponential damage rule postulated by Marco and Starkey [7].

Plots of Miner's coefficient for two-stage loading as given by (3.6) and (3.8) are shown in Fig. 4. It is seen that there is considerable difference between the present prediction and Miner's rule.

We now proceed to the case of multi-stage loadings. The approach will be illustrated by treatment of a three-stage loading. A specimen is subjected to n_1 cycles at amplitude σ_1, then to n_2 cycles at amplitude σ_2. Then the stress level is changed to σ_3 and it is required to determine the remaining lifetime n_3. Consider the two stage loading σ_1, σ_2. Once the stress level is changed to σ_2 we proceed on the damage curve passing through point n_1; σ_1, Fig. 5, until the level σ_2. At this stage the remaining lifetime is $N_2 - n_{12}$. After n_2 cycles are applied at this stress level the remaining lifetime is $N_2 - n_{12} - n_2$. It follows that if the specimen is subjected to $n_{12} + n_2$ cycles at σ_2 the remaining lifetime is the same as that of a specimen subjected to the two stage loading under consideration. (In much vaguer language we would say that the two loadings result in the same damage). Consequently, the three stage loading is equivalent to the two stage loading $\sigma_2 \rightarrow n_{12} + n_2$ cycles, $\sigma_3 \rightarrow n_3$ cycles. On the basis of the equivalent loading postulate we can use the two stage loading procedure which implies that after n_2 cycles at σ_2 we proceed to level σ_3 <u>on the damage curve which passes through the point $n_{12} + n_2$; σ_2.</u> This determines the remaining life-time n_3.

Evidently this procedure can be generalized to multi-stage loadings with any number of stages. It follows that: <u>the damage curves, defined for two-stage loadings, are sufficient information to determine lifetime for any multi-stage loading.</u>

For analytical purposes it is convenient to adopt the following notation: the damage curves which are needed for transition from one loading level to another are denoted D_1, D_2, ..., the last one being the S-N curve. The abscissa of a point on damage curve D_k at stress level σ_ℓ is denoted $n_{k\ell}$. Lifetime at σ_ℓ, i.e. abscissas of points on the S-N curve is denoted N_ℓ.

Consider a three-stage loading, Fig. 5, with damage curves of semi-log type. With the notation adopted

$$n_1 = n_{11}$$

$$n_2 = n_{22} - n_{12} \tag{3.9}$$

$$n_3 = N_3 - n_{23}$$

Since the damage curves have, by hypothesis, the same functional form as the S-N curve, it follows at once that (3.5) can be interpreted as a relation between n_1, n_2 where the damage curve 2 now replaces the S-N curve of the previously considered two level loadings. Therefore

$$(\frac{n_1}{n_{21}})^{\frac{1-s_2}{1-s_1}} + \frac{n_2}{n_{22}} = 1 \tag{3.10}$$

Similarly, we interpret the points $n_{22}; \sigma_2$ and $n_{23}; \sigma_3$ as a two level loading. Therefore, from (3.5)

$$(\frac{n_{22}}{N_2})^{\frac{1-s_3}{1-s_2}} + \frac{n_3}{N_3} = 1 \tag{3.11}$$

Using the functional form of the S-N and damage curves (2.6) and (2.8a), it follows from (3.10 - 11) that

$$[(\frac{n_1}{N_1})^{\frac{1-s_2}{1-s_1}} + \frac{n_2}{N_2}]^{\frac{1-s_3}{1-s_2}} + \frac{n_3}{N_3} = 1 \qquad s_i = \frac{\sigma_i}{\sigma_s} \tag{3.12}$$

Therefore the Miner's coefficient is given by

$$M = 1 + \frac{n_1}{N_1} + \frac{n_2}{N_2} - [\ (\frac{n_1}{N_1})^{\frac{1-s_2}{1-s_1}} + \frac{n_2}{N_2}\ ]^{\frac{1-s_3}{1-s_2}} \tag{3.13}$$

Similar analysis for the log-log forms gives the results

$$[\ (\frac{n_1}{N_1})^{\log s_2/\log s_1} + \frac{n_2}{N_2}\ ]^{\log s_3/\log s_2} + \frac{n_3}{N_3} = 1$$

$$M = 1 + \frac{n_1}{N_1} + \frac{n_2}{N_2} - [(\frac{n_1}{N_1})^{\log s_2/\log s_1} + \frac{n_2}{N_2}]^{\log s_3/\log s_2} \tag{3.14}$$

As a final example for multi-stage loadings, we consider the case of periodic two stage loading where the period (called block in fatigue jargon) is composed of σ_1 for n_1 cycles followed by σ_2 for n_2 cycles. A representation of this loading for nondimensional stresses is shown in Fig. 6a. The loading path in the s-n plane is shown in Fig. 6b.

Let the inverse of a damage curve function $s = f(\gamma, n)$ be written

$$n = g(\gamma, s) \qquad\qquad s = \sigma/\sigma_s \tag{3.15}$$

An alternative form is

$$\gamma = \gamma(s, n) \tag{3.16}$$

With the $n_{k\ell}$ notation adopted above, we proceed in the following manner along the loading path shown in Fig. 6a.

$$n_{11} = n_1$$

$$n_{12} = g(\gamma_1, s_2) \qquad\qquad \gamma_1 = \gamma(n_1, s_1)$$

$$n_{22} = n_{12} + n_2 \qquad\qquad\qquad\qquad\qquad\qquad (3.17)$$

$$n_{21} = g(\gamma_2, s_1) \qquad\qquad \gamma_2 = \gamma(n_{22}, s_2)$$

$$n_{31} = n_{21} + n_1$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

This step-by-step procedure is continued until the S-N curve is reached, thus determining the lifetime for the loading.

The functions g and γ for the two types of damage curves considered are

$$g(s,\gamma) = 10^{\frac{s-1}{\gamma}}$$

$$\gamma(s,n) = \frac{s-1}{\log n} \qquad\qquad\qquad \text{semi-log} \qquad\qquad (3.18)$$

$$g(s,\gamma) = s^{1/\gamma}$$

$$\gamma(s,n) = \frac{\log s}{\log n} \qquad\qquad\qquad \text{log-log} \qquad\qquad (3.19)$$

The procedure is very convenient for numerical computation. It is also easily realized that it can be used to analyze any multi-stage loading.

We now consider the case when the loading amplitude varies in arbitrary continuous fashion with the number of cycles, which is denoted $\bar{n}$. Thus

$$\sigma = \sigma(\bar{n}) \qquad\qquad\qquad\qquad\qquad\qquad (3.20)$$

with inverse

$$\bar{n} = \bar{n}(\sigma) \tag{3.21}$$

The loading (3.20) may be regarded as a multi-stage loading consisting of an infinite number of levels with increments $d\sigma$. Figure 7a shows the loading curve (3.20) with initial value σ_0. The representation of (3.20) in the s-n plane is as follows: At σ_0 plotted on the σ axis proceed $d\bar{n}$ horizontally to the right. Trace a damage curve passing through the point $d\bar{n};\sigma_0$ and on it proceed until $\sigma_0 + d\sigma$, where $d\sigma$ is determined by $d\bar{n}$ and the slope of the loading curve at σ_0. At the point thus reached in the s-n plane, proceed another $d\bar{n}$ horizontally and repeat the previous process. Fig. 7b illustrates this process for a typical point on the loading curve (3.20) and the image of that point in the s-n plane. It is seen that by this procedure the loading curve C is mapped into the curve C' in the s-n plane. Intersection of curve C' with the S-N curve defines failure under the loading program (3.20).

This procedure of construction of the image C' of C will now be expressed analytically. Let the equation of C' be denoted

$$G(s,n) = 0 \qquad\qquad s = \sigma/\sigma_s \tag{3.22}$$

The relation between $d\sigma$ and dn on this curve is illustrated in Fig. 7b. The differential dn is given by

$$dn = dn_D + d\bar{n} \tag{3.23}$$

where dn_D is the contribution due to movement from σ to $\sigma + d\sigma$ on the damage curve which passes through the point $n_D;\sigma$. Note that at this point

$$n_D = n \tag{3.24}$$

Let the damage curve equation (2.5) be written in the form

$$n_D = g(s,\gamma) \tag{3.25}$$

Since γ is constant on this curve it follows that

$$dn_D = \frac{\partial g}{\partial s} \, ds \tag{3.26}$$

In view of (3.24 - 25), γ of the damage curve passing through the point $n_D; \sigma$ can be expressed from (3.20) in the form

$$\gamma = \gamma(s,n) \tag{3.27}$$

Now write (3.21) in the form

$$\bar{n} = h(s) \tag{3.28}$$

Introducing (3.26 - 3.28) into (3.23) we obtain

$$dn = [\frac{\partial g(s,\gamma)}{\partial s} + h'(s)]ds \tag{a}$$

$$\gamma = \gamma(s,n) \tag{b}$$

(3.29)

where (b) is to be introduced into (a) after the partial differentiation has been performed.

Equ.(3.29) is a nonlinear ordinary first order differential equation of the type

$$dn = \phi(s,n)ds$$

Its integration starting out from an initial point $n_o ; \sigma_o$ gives the equation of the curve C', (3.22).

The intersection of (3.22) with the S-N curve defines the stress level σ_u at which failure occurs under the loading program (3.20). The cycle ordinate n in the S-N plane associated with the intersection point does not define the lifetime. The lifetime $\bar{N}_u$ is defined from (3.20) by (see Fig. 7)

$$\bar{N}_u = \bar{n}(\sigma_u) \tag{3.30}$$

This raises a problem when there is no one-to-one correspondence between σ and $\bar{n}$ as defined by (3.20), which happens when (3.20) is not a monotonic function. In this event it is necessary to determine during the integration of (3.29) the images of the extrema of (3.20) on C'. In this fashion σ_u is then located on the proper monotonic part of (3.20).

In the important case of load discontinuities in (3.20) it is necessary to integrate (3.29) until the image of the discontinuity in the s-n plane is reached. The jump in σ then proceeds on the damage curve which passes through the end point of integration reached, in the s-n plane. Then the integration is continued until the next discontinuity, etc.

The functions $g(\gamma,s)$ and $\gamma(s,n)$ which appear in (3.29) have been given for semi-log and log-log damage curves by (3.18 - 19). Therefore, for these cases, the differential equation (3.29) assumes the forms

$$dn = [\;\frac{n \log n}{s - 1}\;\; \ell n\; 10 + h'/s)]ds \qquad (a) \qquad \text{semi-log}$$

$$(3.31)$$

$$dn = [\;\frac{n \log n}{s \log s} + h'(s)]ds \qquad (b) \qquad \text{log-log}$$

A representation of (3.32) in terms of the variables $n, \bar{n}$ can be obtained by writing (3.20) in the form

$$s = H(\bar{n}) \tag{3.32}$$

and substituting into (3.21).

It appears that analytical integration of equs. (3.31) is not possible but the equations are readily integrated numerically by use of the Runge Kutta method, for example. For this purpose it is convenient to introduce the variable

$$\eta = \log n \tag{3.33}$$

Then (3.31) assume the forms

$$\frac{d\eta}{ds} = \frac{\eta}{s-1} + \frac{h'(s)}{10^\eta \ell n\; 10} \qquad (a)$$

$$(3.34)$$

$$\frac{d\eta}{ds} = [\;\frac{\eta}{s \log s} + \frac{h'(s)}{10^\eta}\;]\;\frac{1}{\ell n\; 10}. \qquad (b)$$

As an example we consider the loading program shown in Fig. 8a. The initial amplitude is given by $s_o = \sigma_o/\sigma_s$ and then diminishes linearly to a value s_1. Cycling is then continued at constant amplitude until failure. The linear variation of cycling amplitude is described by

$$s = s_0 + (s_1 - s_0) \frac{\bar{n}}{n_0} \qquad (3.35)$$

It follows from the definition (3.28) and from (3.35) that

$$h'(s) = \frac{n_0}{s_1 - s_0} \qquad (3.36)$$

With this value of $h'(s)$ equs. (3.34) must be integrated starting out at the initial point $\eta = 0$; $s = s_0$. Note that $\eta = 0$ corresponds to $n = 1$, not $n = 0$. This is an insignificant correction which is required because of the log representation.

Integration of the differential equations gives the image C' of the loading function in s-n plane. This image is shown in Fig. 8b for the semi-log case, equ. (3.34a) when $s_0 = 0.7$, $n_0 = 10^4$. It is seen that the image C' in the s-n plane of the linear loading curve C starts out horizontally and then approaches asymptotically a straight line passing through the point $\eta = 0$, $s = 1.0$, i.e., a damage curve. This is a general characteristic of solutions of (3.34) since for large η the second term on the right sides becomes very small in comparison to the first terms because of 10^η in the denominator. If the second term is neglected the solution of the differential equations are the damage curves (2.8a), (2.9a) respectively. This is also evident from (3.23) since neglect of the second term on the right side of (3.34) is equivalent to neglecting $d\bar{n}$ in comparison to dn_D.

For linear loading from s_0 to s_1 the remaining lifetime is given by the horizontal distance between the point on C' where $s = s_1$, and the S-N curve, in s,n coordinates. Note that the present figure is in s,log n coordinates. Fig. 8b shows remaining lifetimes for various values of s_1.

In order to compare the present prediction to those of Miners' rule it is necessary to write Miners' rule for a continuous loading function. To do this the continuous loading (3.20) is interpreted as a multi-stage loading with successive load increments $d\sigma$ and cycle increments $d\bar{n}$. Then (1.3) may be written

$$\int_{\sigma_0}^{\sigma_u} \frac{d\bar{n}}{N(\sigma)} = \int_{\sigma_0}^{\sigma_u} \frac{d\bar{n}/d\sigma}{N(\sigma)} \, d\sigma = 1 \tag{3.37}$$

where $N(\sigma)$ is lifetime at constant σ as given by the S-N curve. Equ. (3.38) defines the failure stress σ_u. Lifetime $\bar{N}_u$ is then given by (3.30).

In terms of the nondimensional stress $s = \sigma/\sigma_s$ (3.37) assumes the form

$$\int_{s_0}^{s_u} \frac{h'(s)}{N(s)} \, ds = 1 \tag{3.38}$$

For semi-log and log-log representations (2.6), (2.7) we have respectively

$$N(s) = 10^{\frac{s-1}{\Gamma}}$$

$$N(s) = s^{1/\Gamma} \tag{3.39}$$

while $h'(s)$ for the linear loading is given by (3.36). For the present loading (3.38) assumes the form

$$\int_{s_0}^{s_1} \frac{h'(s)}{N(s)} \, ds + \frac{n_1}{N(s_1)} = 1 \tag{3.40}$$

which defines the residual lifetime n_1 on the basis of Miner's rule.

It follows from (3.40), (3.36) and (3.39) that n_1 is given by

$$n_1 = N(s_1) - \frac{\Gamma}{\ell n\ 10}\ \frac{n_0}{s_0 - s_1}\ [1 - \frac{N(s_0)}{N(s_1)}\ ] \qquad \text{(a) semi-log}$$

$$\text{(3.41)}$$

$$n_1 = N(s_1) - \frac{\Gamma}{\Gamma - 1}\ \frac{n_0}{s_0 - s_1}\ [s_0\ \frac{N(s_1)}{N(s_0)} - s_1] \qquad \text{(b) log-log}$$

The n_1 predicted by (3.41a) are also shown in Fig. 8b. It is seen that there are significant differences between remaining lifetimes as predicted by Miner's rule and the present theory.

Analysis has also been performed for the case of pure linear variation of cycle amplitude (increasing and decreasing) with number of cycles, until failure. Results obtained for various such cases showed little numerical difference between lifetime predictions by present theory and by Miner's rule. It is of great importance to arrive at some general conclusions with respect to loading characteristics for which Miner's rule and the new theory predict significantly different lifetimes. It appears that much further analytical work is needed to obtain such information.

4. Summary and Conclusion

A rational phenomenological theory to compute fatigue lifetime under arbitrary cyclic loading programs has been developed. The theory is based on the concept of damage curve families which represent all possible two stage cyclic loadings - to - failure. The damage curves may be obtained experimentally. In the present work their analytical form is postulated on the basis of assumed similarity with S-N curve analytical representations.

The theory is based on a basic assumption which has been termed the equivalent loading postulate which defines in macroscopic terms specimens which have suffered the same damage under different loading programs. This postulate leads to uniqueness of the damage curves and underlies methods of lifetime analysis in terms of the damage curves.

Procedures of lifetime prediction have been given for piecewise constant cycle amplitude variation (multi-stage loading) as well as for continuous variation of cycle amplitude with number of cycles. In the first case analysis consists of a simple step by step procedure in terms of damage curves. In the second case solution of initial value problems for first order nonlinear differential equations is required.

Damage curve families considered in this work are straight lines in $\log n$, σ or in $\log n$, $\log \sigma$ planes.

Analytical results have shown that for multi-stage loading programs the predictions of the present theory are considerably different than those based on Miner's rule. Analysis of a few cases of continuous amplitude variation programs has shown that in certain cases present predictions are significantly different from those based on Miner's rule while in others this is not the case.

Further analytical work is needed to search for loading characteristics which lead to significant discrepancies between prediction by Miner's rule and present theory. It is also of great importance to establish an analytical procedure for random loadings, i.e., where the cycle amplitude is a random variable and only statistical loading information is available. This is a difficult problem since it requires solution of nonlinear differential equations with random inputs.

An experimental program is now under way to examine the validity of the theory. This consists of two stage loadings-to-failure of gloss fiber/epoxy specimens.

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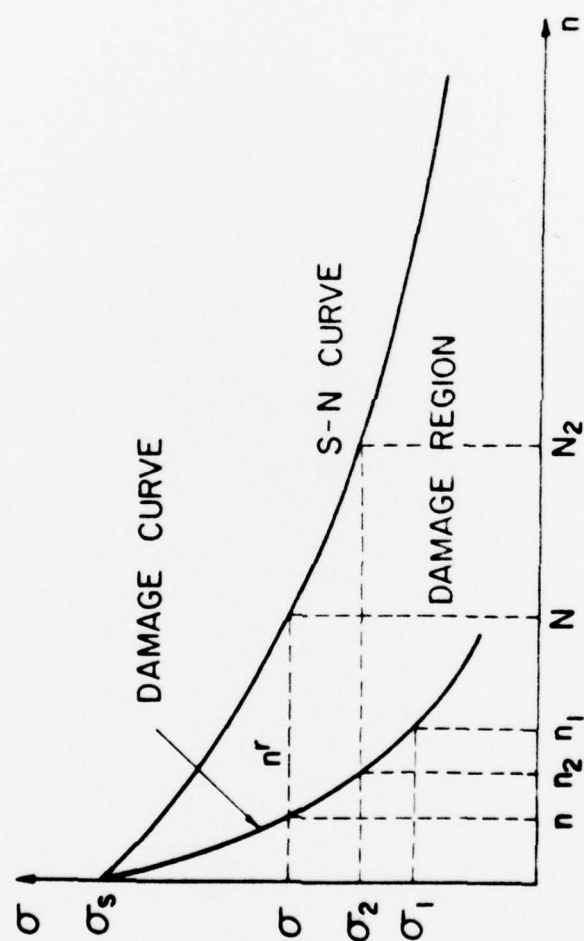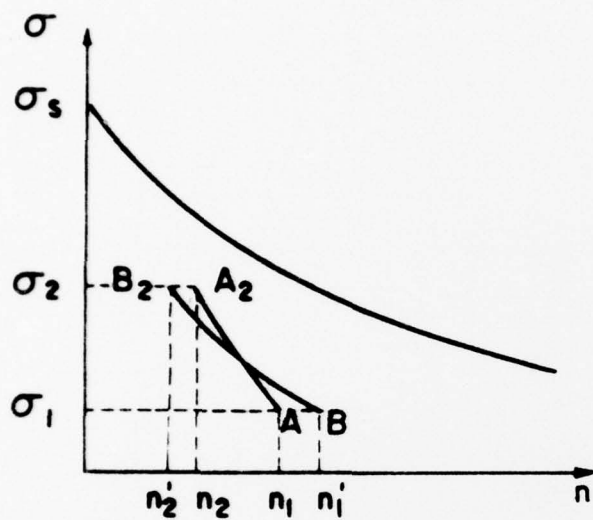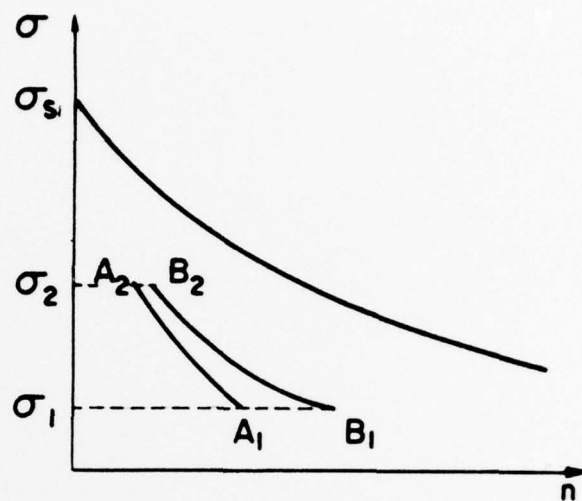
FIG. 1   DAMAGE CURVE ;   DAMAGE REGION
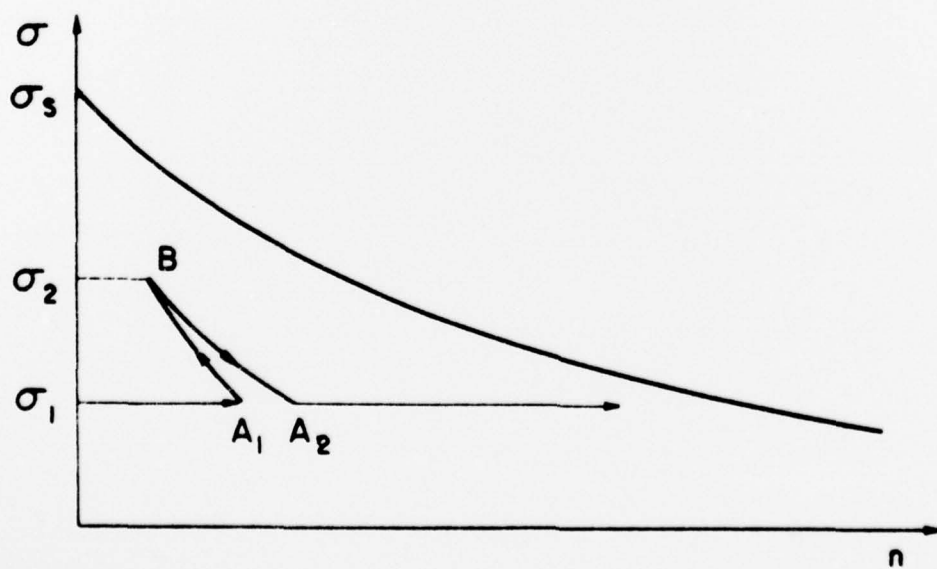
FIG. 2 NON-INTERSECTION OF DAMAGE CURVES
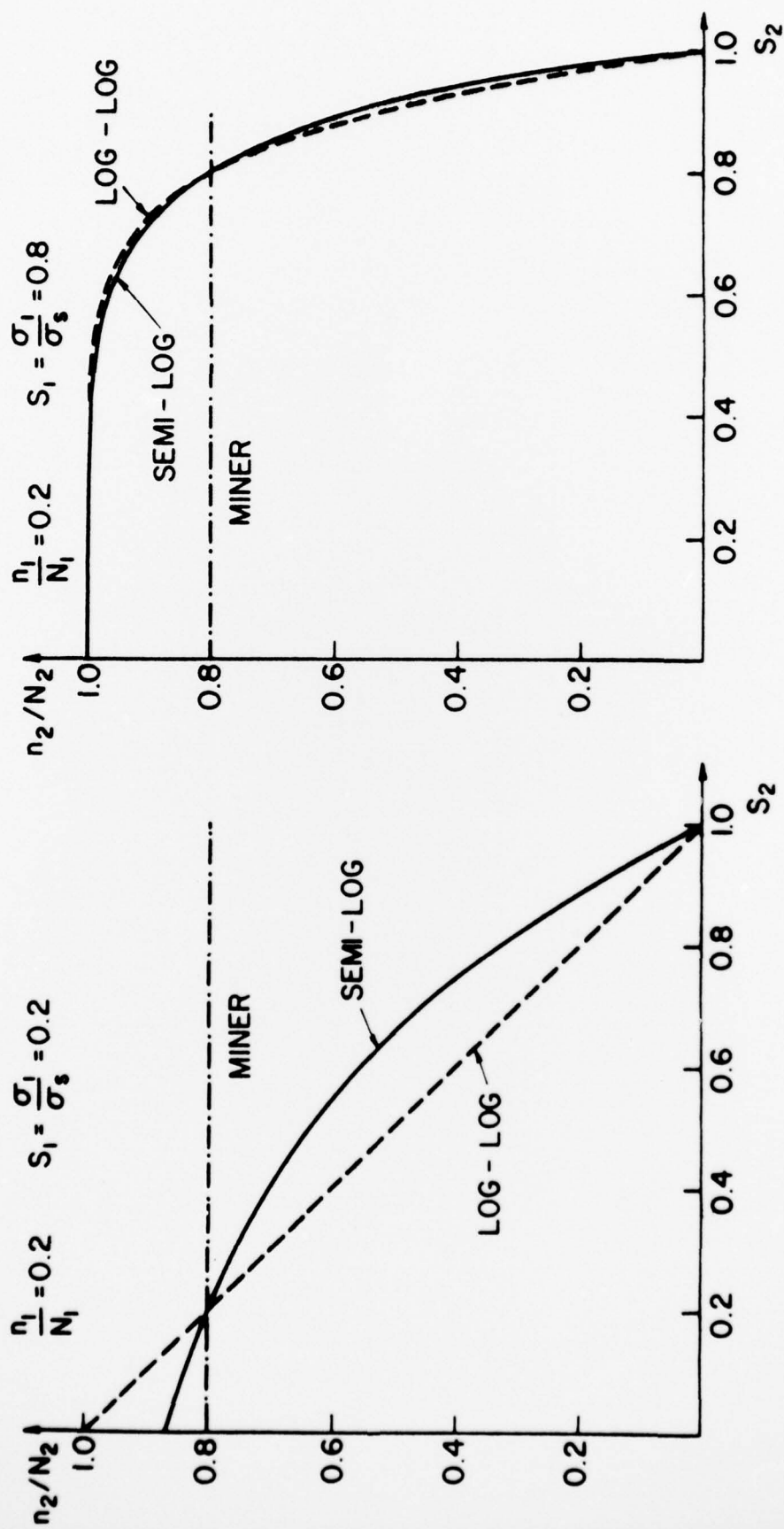

FIG. 3 HYSTERESIS EFFECT ON DAMAGE CURVES
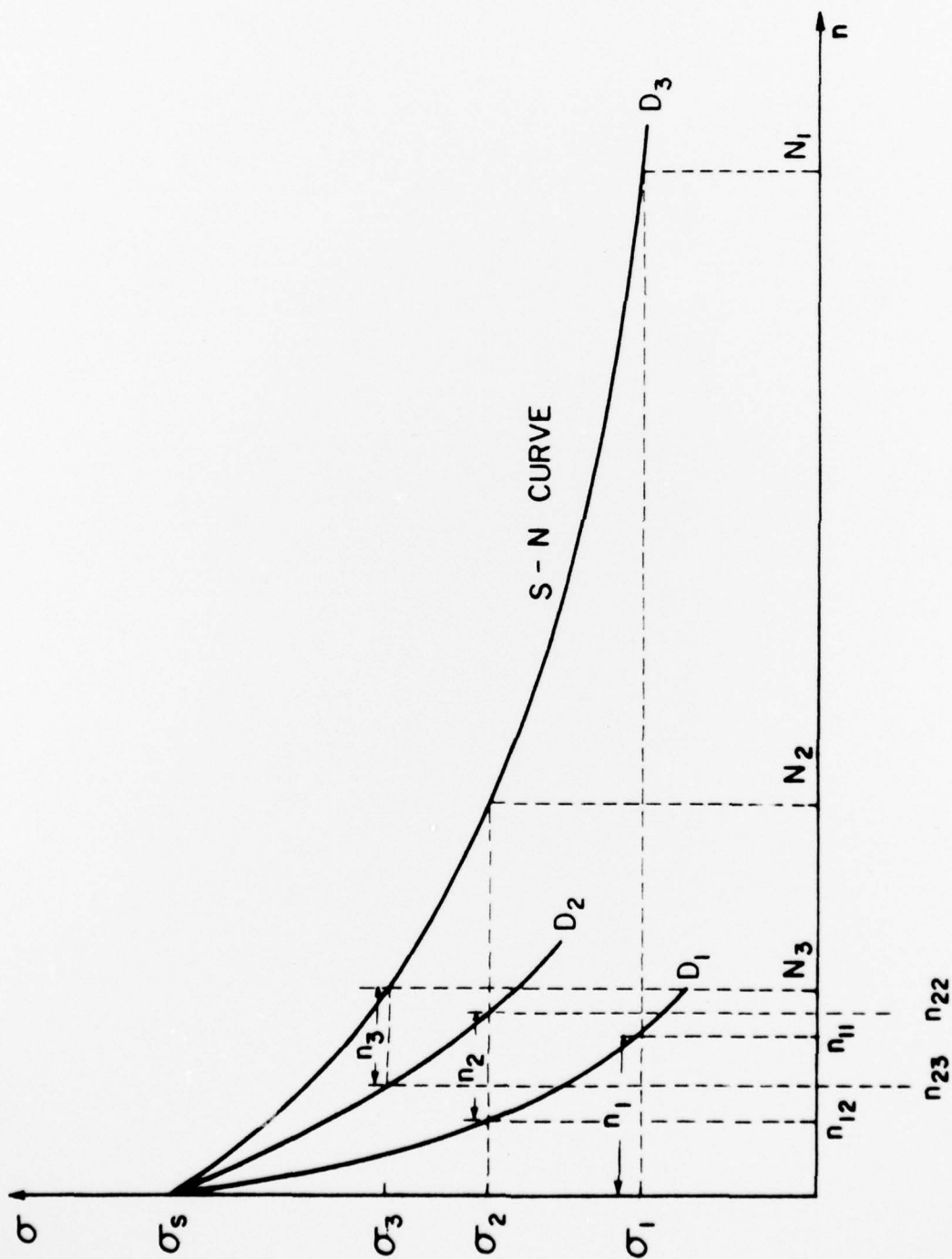
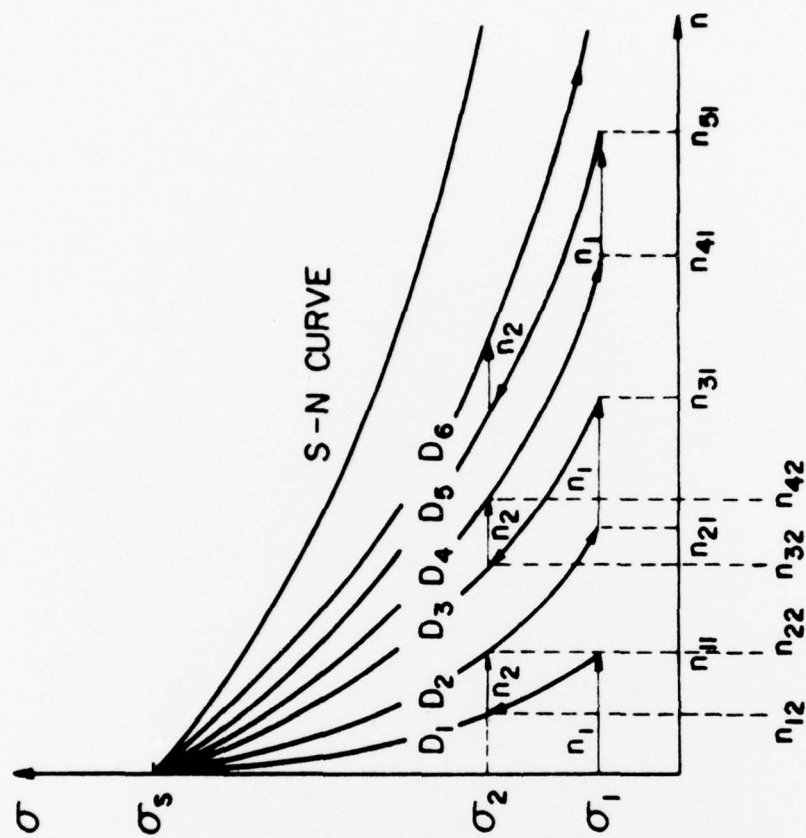FIG. 4   LIFE-TIME PREDICTIONS   TWO STAGE LOADING

S – N CURVE

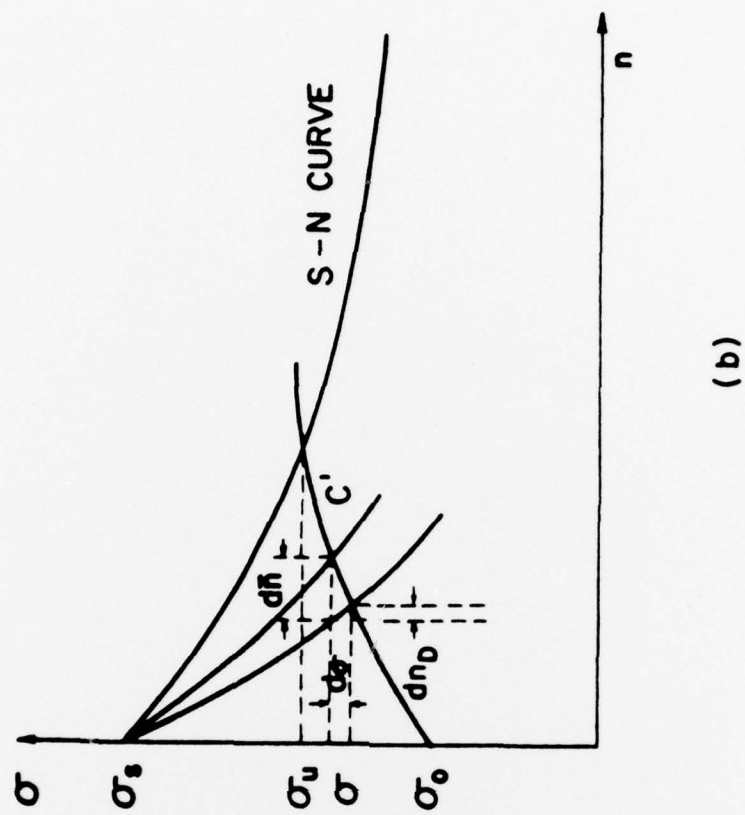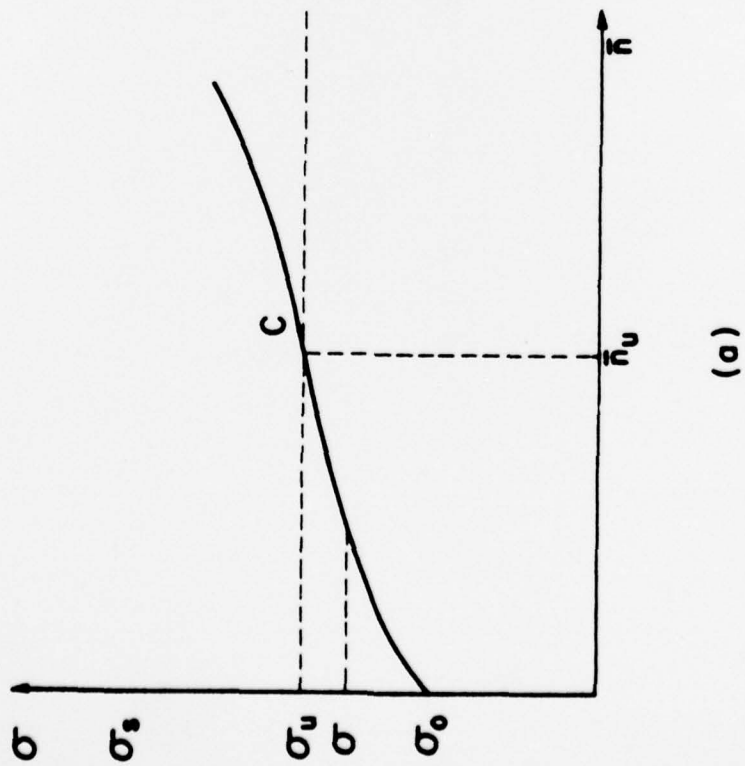THREE LEVEL LOADING

FIG. 5

FIG. 6   TWO STAGE BLOCK LOADING

FIG. 7   LOADING WITH CONTINUOUS VARIATION OF CYCLE AMPLITUDE

VARIATION OF CYCLE AMPLITUDE

$s = s_o + (s_1 - s_o)\bar{n}/n_o$

$n_o = 10^4$

$s_o = 0.7$

$s_1 = 0.3$

C

FAILURE

$\bar{n}$

n_1

s

1.0

FIG. 8a   BILINEAR LOADING PROGRAM

MINER

$n_1 = 4420$

$n_1 = 38,600$

$n_1 = 380,000$

$n_1 = 3,797 \times 10^6$

$n_1 = 5370$

$n_1 = 54,200$ C'

$n_1 = 608,000$

$n_1 = 6,649 \times 10^6$

$s_1 = 0.6$

$s_1 = 0.5$

$s_1 = 0.4$

$s_1 = 0.3$

$s = 1 - \log n / 9.3217$

LIMIT DAMAGE CURVE

$s = 1 - \log n / 10$

S/N CURVE

log n

s

1.0

0.8

0.6

0.4

0.2

0   2   4   6   8   10

FIG. 8b   RESIDUAL LIFETIME ANALYSIS